Topic — Problems based on partial derivatives of a function of two independent variables

Class — B.Sc I (Hons and Sub) Date — 26-10-2021

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Important formulae

(i) If $u = f(x, y)$

$$\Rightarrow du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

(ii) If $u = f(x, y, z)$

$$\text{then } du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

(iii) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ Where u is the homogeneous function of two independent variables x and y of degree n.

(iv) $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f$

Problem 1.

If $u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$

Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Here. By the question.

$$u = \begin{vmatrix} x^2 & y^2 & z^2 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

Operating along 1st Row

$$u = x^2\begin{vmatrix} y & z \\ 1 & 1 \end{vmatrix} - y^2\begin{vmatrix} x & z \\ 1 & 1 \end{vmatrix} + z^2\begin{vmatrix} x & y \\ 1 & 1 \end{vmatrix}$$

$$= x^2(y-z) - y^2(x-z) + z^2(x-y)$$

$$= x^2(y-z) + y^2(z-x) + z^2(x-y) \quad \text{--- (I)}$$

Now Diff. partially with respect to x, y and z respectively

$$\frac{\partial u}{\partial x} = 2x(y-z) - y^2 + z^2$$

$$\frac{\partial u}{\partial y} = x^2 + 2y(z-x) - z^2 \quad \text{(ii)}$$

$$\frac{\partial u}{\partial z} = -x^2 + y^2 + 2z(x-y) \quad \text{(iii)}$$

On adding

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}$$

$$= 2x(y-z) - \cancel{y^2} + \cancel{z^2} + \cancel{x^2} + 2y(z-x) - \cancel{z^2} - \cancel{x^2} + \cancel{y^2} + 2z(x-y)$$

$$= 2[\cancel{xy} - \cancel{xz} + \cancel{yz} - \cancel{xy} + \cancel{zx} - \cancel{yz}]$$

$$= 0$$

Problem 2.

If. $u = a\sin\frac{x}{y} + b\cos y/x$

Prove that-

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Here By the question

$u = a\sin\frac{x}{y} + b\cos y/x$ —— (1)

Diff. partially with respect to x

$$\frac{\partial u}{\partial x} = a\cos\frac{x}{y}\cdot\frac{1}{y} - b\sin y/x\,(-y/x^2)$$

$$= \frac{a}{y}\cos\frac{x}{y} + \frac{by}{x^2}\sin\frac{y}{x}$$

now

$$x\frac{\partial u}{\partial x} = \frac{xa}{y}\cos\frac{x}{y} + \frac{bxy}{x^2}\sin y/x$$

$$= \frac{x}{y}a\cos\frac{x}{y} + \frac{by}{x}\sin y/x$$

Again differentiate (1) partially with respect to y

$$\frac{\partial u}{\partial y} = -\frac{1}{y^2}ax\cos\frac{x}{y} - b\sin y/x\cdot\frac{1}{x}$$

$$\frac{\partial u}{\partial y} = -\frac{a}{y^2}\cos\frac{x}{y} - \frac{b}{x}\sin y/x$$

$$y\frac{\partial u}{\partial y} = -\frac{ax}{y}\cos\frac{x}{y} - \frac{by}{x}\sin y/x$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{y}a\cos\frac{x}{y} + \frac{by}{x}\sin y/x - \frac{ax}{y}\cos\frac{x}{y} - \frac{by}{x}\sin y/x$$

$= 0$ Hence the required Result

Topic - Homogeneous Differential Equation of First order and First Degree.

Class - B.Sc II (Math- HONS) (Group B)

Date -

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Study Material - Homogeneous Equations -

A Differential Equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

Where $f(x,y)$ and $g(x,y)$ are homogeneous function of x and y of the same degree is called Homogeneous Differential Equation

Working Rule STEP I - put $\frac{y}{x} = v$

$$\Rightarrow y = vx$$

STEP II - Diff with respect to x

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

STEP III - put $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and $y = vx$ in the Given Equation

On doing so, the given Differential Equation can be reduced to the form in which the variables are separable.

Problem :→ 1

Solve $x^2y\,dx - (x^3+y^3)dy = 0$

Solution :→ The Given Equation is

$$x^2y\,dx - (x^3+y^3)dy = 0$$

$$\Rightarrow (x^3+y^3)dy = x^2y\,dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \quad \text{——} \; (1)$$

put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1)

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3x^3}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v}{1+v^3}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v}{1+v^3} - v$$

$$x\frac{dv}{dx} = \frac{\cancel{v} - \cancel{v} - v^4}{1+v^3}$$

$$\Rightarrow \frac{dx}{x} = -\frac{1+v^3}{v^4}dv$$

$$\int\frac{dx}{x} = \int\left[-v^{-4} - \frac{1}{v}\right]dv$$

$$\log x = -\frac{v^{-3}}{[illegible]} - \log v =$$

$$\Rightarrow \log x = \frac{1}{3v^3} - \log v + c$$

$$\log x = \frac{x^3}{3y^3} - \log\frac{y}{x} + c$$

$$\log x = \frac{x^3}{3y^3} - \log y + \log x + c$$

$$\Rightarrow \log y = \frac{x^3}{3y^3} + c$$

Which is the required Solution

Ex:→ Solve.

$$x\,dy - y\,dx - \sqrt{x^2+y^2}\,dx = 0$$

Solution →

⇒ the Given differential Equation is

$$x\,dy - y\,dx - \sqrt{x^2+y^2}\,dx = 0$$

$$x\,dy = \left[y + \sqrt{x^2+y^2}\right]dx$$

$$\frac{dy}{dx} = \frac{y + \sqrt{x^2+y^2}}{x} \quad \text{——— (1)}$$

Put $y/x = v \Rightarrow y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

putting these values in ①
we have

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x} =$$

$$\Rightarrow v + x\frac{dv}{dx} = v + \sqrt{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \sqrt{1+v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{dv}{\sqrt{1+v^2}}$$

On integrating

$$\int \frac{dx}{x} = \int \frac{dv}{\sqrt{1+v^2}}$$

$$\Rightarrow \log x = \log(v + \sqrt{1+v^2}) + \log c$$

$$\Rightarrow \log x = \log c(v + \sqrt{1+v^2})$$

$$\therefore x = c(v + \sqrt{1+v^2})$$

putting $v = y/x$

$$= c\left(\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}\right)$$

$$= c\left(\frac{y + \sqrt{x^2+y^2}}{x}\right)$$

$$\Rightarrow x^2 = c(y + \sqrt{x^2+y^2})$$

which is the required solution

Problem Solve

$$(x^2-y^2)\frac{dy}{dx} = 2xy$$

Solution – The Given Equation is

$$(x^2-y^2)\frac{dy}{dx} = 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy}{x^2-y^2} \quad \text{——— (1)}$$

Put $y = v.x$ in (1)

and $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

We have

$$v + x\frac{dv}{dx} = \frac{2x \cdot vx}{x^2 - v^2x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{2v}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v}{1-v^2} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{2v - v + v^3}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v + v^3}{1-v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v(1+v^2)}{1-v^2}$$

$$\Rightarrow \frac{dx}{x} = \frac{1-v^2}{v(1+v^2)}dv$$

On integrating

$$\frac{dx}{x} = \int \frac{1-v^2}{v(1+v^2)}dv$$

$\Rightarrow \int \frac{dx}{x} = \int \left(\frac{1}{v} - \frac{2v}{1+v^2} \right) dv$

$\Rightarrow \log x = \log v - \log(1+v^2)$

$\Rightarrow \log x = \log \frac{y}{x} - \log\left(1+\frac{y^2}{x^2}\right)$ put $v = y/x$

$\Rightarrow \log x = \log y - \log x - \log(x^2+y^2) + 2\log x + k$

$\Rightarrow \log x = \log y - \log(x^2+y^2) + \log x + k$

$\Rightarrow \log y = \log(x^2+y^2) + \log c$ put $k = \log c$

$\log y = \log c(x^2+y^2)$

$\therefore y = c(x^2+y^2)$

which is the required solution.